5. Kupradze, V. D., Boundary Value Problems of the Theory of Oscillations and Integral Equations. Moscow-Leningrad, Gostekhizdat, 1950.
6. Kupradze, V. D., Potential Methods in the Theory of Elasticity. Moscow, Fizmatgiz, 1963.
7. Sherman, D. I., On certain problems of the theory of steady oscillations. Izv. Akad. Nauk SSSR, Ser. matem., Vol. 9, № 4, 1945.
8. Watson, G. N., A Treatise of the Theory of Bessel Functions. Cambridge Univ. Press, 1948.

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INVERSION OF RELATIONS OF THE THEORY OF PLASTIC FLOW FOR HARDENING BODIES

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The inversion of the fundamental relationships of the theory of plastic flow of hardening bodies is obtained in the neighborhood of a regular point of an arbitrary loading surface. The stress increments are consequently expressed explicitly in terms of the strain increments.

The fundamental relationships of the theory of a plastic hardening body [1, 2] under the assumption of the existence of loading functions are in the form of relationships expressing the increments of strain in terms of the increments of stress. Upon formulating the problems in displacements, for example in the case of three-dimensional stability problems [3, 4], the increments in stress must be expressed in terms of the increments of strain, i.e. the fundamental relationships must be inverted. Such an inversion is realized below in the neighborhood of a regular point of an arbitrary loading surface for an isothermal strain process in the case of small strains.

1. Following [1, 2], let us write the fundamental relationships of the theory of a plastic hardening body in the neighborhood of a regular point of the loading surface. We represent the total strain increment as the sum of increments in the elastic and plastic strains (we introduce the compliance tensor C for the elastic strain, and we proceed from the associated flow law for plastic deformation)

$$d\varepsilon_{nm} = d\varepsilon_{nm}{}^e + d\varepsilon_{nm}{}^p \tag{1.1}$$

$$d\varepsilon_{nm}^e = C_{nmij}\, d\sigma^{ij} \tag{1.2}$$

$$d\varepsilon_{nm}^p = d\lambda \frac{\partial f}{\partial \sigma^{mn}}, \quad \text{when} \quad f = 0,\ df = 0 \quad \text{and} \quad d'f > 0 \tag{1.3}$$

$$d\varepsilon_{nm}^p = 0, \ \text{when} \ \ f = 0 \quad \text{and} \quad df \equiv d'f \leqslant 0 \quad \text{or} \quad f < 0$$

where f denotes the loading function; the equation of the loading surface is hence

$$f(\sigma^{ij}, g_{ij}, \varepsilon_{ij}{}^p, \chi_s, k_s) = 0 \tag{1.4}$$

Here g_{ij} are the covariant components of the metric tensor of a Lagrange coordinate system in the undeformed state, σ^{ij} are the contravariant components of the stress tensor, ε_{ij}^{p} are the covariant components of the plastic strain tensor, χ_s are the hardening parameters which can be related to the residual strains by nonholonomic constraints, and k_s are constants of the material.

The differential dependences for the hardening parameters can be represented for a loading in one of the forms

$$d\chi_s = A_s^{ij} d\varepsilon_{ij}^{p}, \quad A_s^{ij} = A_s^{ij}(\sigma^{nm}, \varepsilon_{nm}^{p}) \tag{1.5}$$

$$d\chi_s = B_s d'f \equiv B_s \frac{\partial f}{\partial \sigma^{ij}} d\sigma^{ij}, \quad B_s = B_s(\sigma^{nm}, \varepsilon_{nm}^{p}) \tag{1.6}$$

The expressions (1.1)—(1.6) are fundamental in the theory of plastic hardening bodies.

2. The fundamental relationships can also be represented in a somewhat different form. To this end, we proceed as follows. From (1.3), (1.4) and (1.6) we determine the factor $d\lambda$ and substitute it into (1.3). We consequently obtain from (1.1)—(1.3)

$$d\varepsilon_{nm} = (C_{nmij} + K_{nmij}) d\sigma^{ij}, \quad \text{when } f = 0,\ df = 0 \text{ and } d'f > 0 \tag{2.1}$$

$$d\varepsilon_{nm} = C_{nmij} d\sigma^{ij}, \quad \text{when } f = 0 \text{ and } df \equiv d'f \leqslant 0 \quad \text{or } f < 0$$

$$K_{nmij} = -\left[\frac{\partial f}{\partial \sigma^{\alpha\beta}}\left(\frac{\partial f}{\partial \varepsilon_{\alpha\beta}^{p}} + A_s^{\alpha\beta}\frac{\partial f}{\partial \chi_s}\right)\right]^{-1} \frac{\partial f}{\partial \sigma^{nm}} \frac{\partial f}{\partial \sigma^{ij}} \tag{2.2}$$

$$K_{nmij} = -\left(1 + B_s \frac{\partial f}{\partial \chi_s}\right)\left(\frac{\partial f}{\partial \varepsilon_{\alpha\beta}^{p}} \frac{\partial f}{\partial \sigma^{\alpha\beta}}\right)^{-1} \frac{\partial f}{\partial \sigma^{nm}} \frac{\partial f}{\partial \sigma^{ij}} \tag{2.3}$$

Let us introduce the tensor K, whose covariant components are determined from (2.2) in the case of (1.5), and from (2.3) in the case of (1.6), into (2.1). As a result of the inversion, (2.1) can be represented as

$$d\sigma^{ij} = E_p^{ijnm} d\varepsilon_{nm}, \quad \text{when } f = 0,\ df = 0 \text{ and } d'f > 0 \tag{2.4}$$

$$d\sigma^{ij} = E^{ijmn} d\varepsilon_{nm}, \quad \text{when } f = 0 \text{ and } df \equiv d'f \leqslant 0 \quad \text{or } f < 0$$

$$E_p^{kqnm}(C_{nmij} + K_{nmij}) = g_i^{\ k} g_j^{\ q} \tag{2.5}$$

$$E^{kqnm} C_{nmij} = g_i^{\ k} g_j^{\ q} \tag{2.6}$$

Let us note that the tensor E, whose contravariant components are evaluated as a result of solving the system (2.6), is an elastic modulus tensor (of a linear anisotropic elastic body), whose representation for different classes of anisotropy is presented in known courses of the theory of elasticity. Hence, the question of the inversion of the relationships (2.1) for the known tensor E reduces to determining the contravariant components of the tensor E_p as a result of solving the system of equations (2.5) under conditions (2.6).

The tensors E, C, K and E_p have the following properties:

$$E^{kqnm} = E^{qknm} = E^{kqmn} = E^{nmkq}, \quad C_{nmij} = C_{mnij} = C_{nmji} = C_{ijnm} \tag{2.7}$$

$$K_{nmij} = K_{mnij} = K_{nmji} = K_{ijnm}, \quad E_p^{kqnm} = E_p^{qknm} = E_p^{kqmn} = E_p^{nmkq}$$

3. Let us turn to the evaluation of components of the tensor E_p. We note that there follows from (2.2) and (2.3) that the components of the tensor K can be defined as

$$K_{nmij} = -M_{nm}M_{ij} \tag{3.1}$$

$$M_{nm} = \left[\frac{\partial f}{\partial \sigma^{\alpha\beta}}\left(\frac{\partial f}{\partial \varepsilon^p_{\alpha\beta}} + A_s^{\alpha\beta}\frac{\partial f}{\partial \chi_s}\right)\right]^{-1/2}\frac{\partial f}{\partial \sigma^{nm}} \tag{3.2}$$

$$M_{nm} = \left[\left(1 + B_s\frac{\partial f}{\partial \chi_s}\right)\left(\frac{\partial f}{\partial \varepsilon^p_{\alpha\beta}}\frac{\partial f}{\partial \sigma^{\alpha\beta}}\right)^{-1}\right]^{1/2}\frac{\partial f}{\partial \sigma^{nm}} \tag{3.3}$$

(the components of the tensor M in (2.2) have the form (3.2), and in (2.3) have the form (3.3)).

Introducing the notation (3.1), we obtain a system of equations (2.6) from (2.5) to determine the contravariant components of the tensor E_p. We seek the solution of this same system as

$$E_p^{kqnm} = E^{kqnm} + zE^{kqt_1t_2}M_{t_1t_2}E^{nmt_3t_4}M_{t_3t_4} \tag{3.4}$$

where z is an unknown scalar function. Substituting (3.4) into (2.5) and taking (3.1), (2.6) and the properties (2.7) of the tensors E and C here, we obtain

$$E^{kqnm}M_{nm}M_{ij}\,[1 - z\,(1 - E^{t_1t_2t_3t_4}M_{t_1t_2}M_{t_3t_4})] = 0$$

after a number of manipulations.

It hence follows that

$$z = (1 - E^{t_1t_2t_3t_4}M_{t_1t_2}M_{t_3t_4})^{-1} \tag{3.5}$$

We note that the solution (3.4), (3.5) is meaningless for those strain processes for which the conditions of disappearance of the expression in the parentheses in (3.5) are satisfied. Taking (3.1) into account, these conditions can be given the following form:

$$E^{kqnm}K_{nmkq} = -1 \tag{3.6}$$

We analyze the case when conditions (3.6) can be satisfied. It follows from (2.1) - (2.3) that the tensor K characterizes the increment in the plastic strains of a hardening body, and it follows from the conditions (3.6) that the components of the tensor K must be expressed only in terms of the tensor E which characterizes the increments in the elastic strains. Therefore, a solution in the form of (3.4), (3.5) is meaningless only when the components of the tensor K characterizing the increments in the plastic strains in a hardening body are expressed only in terms of components of the elastic modulus tensor E of the same body independently of the nature of the simplification and equivalently, of the magnitude of the stresses and plastic strains.

This case apparently holds only for individual strain processes which are not characteristic for hardening bodies. Thus, we obtain from a comparison between (2.6) and (3.6) that the conditions (3.6) will be satisfied when

$$K_{nmkq} = -C_{nmkq} \tag{3.7}$$

It follows from (1.1), (1.2) and (3.1) and the first expression in (2.1) that under the active loading the increments in the plastic strains are equal in magnitude to the corresponding increments in the elastic strains and oppposite in sign; the increments in the total strains hence equal zero. Such a strain process is apparently meaningless.

Consequently, in the general case conditions (3.6) are not satisfied for plastic hardening bodies, but the components of the tensor E_p in (2.4) can be represented, according to (3.4) and (3.5), as follows:

$$E_p^{ijmn} = E^{ijnm} - E^{ijt_1t_2}M_{t_1t_2}E^{nmt_3t_4}M_{t_3t_4}(E^{\alpha\beta\gamma\delta}M_{\alpha\beta}M_{\gamma\delta} - 1)^{-1} \qquad (3.8)$$

Therefore, when the loading function is given in the form (1.4), then the fundamental relationships of the theory of plastic flow of hardening bodies can be represented in one of the following two forms in the neighborhood of a regular point of the loading surface. The first form is like (2.1) taking (3.1) into account; in a somewhat different notation this form has been examined in [1, 2], etc. The second form, which is an inversion of the first, can be represented according to (2.4) and (3.8), as follows:

$$d\sigma^{ij} = (E^{ijnm} - E_*^{ijnm})\, d\varepsilon_{nm}, \quad \text{when} \quad f = 0, \ df = 0 \ \text{and} \ d'f > 0 \qquad (3.9)$$

$$d\sigma^{ij} = E^{ijnm}\, d\varepsilon_{nm}, \quad \text{when} \quad f = 0 \ \text{and} \ df \equiv d'f \leqslant 0 \quad \text{or} \quad f < 0$$

The tensor E_*, whose contravariant components are determined from the expressions

$$E_*^{ijnm} = E^{ijt_1t_2}M_{t_1t_2}E^{nmt_3t_4}M_{t_3t_4}(E^{\alpha\beta\gamma\delta}M_{\alpha\beta}M_{\gamma\delta} - 1)^{-1} \qquad (3.10)$$

has been introduced into (3.9). The components of the tensor M in (2.1) and (3.1), as well as in (3.9) and (3.10), are determined from the expressions (3.2) if the differential dependences for the hardening parameters have the form (1.5), and from (3.3) if these dependences have the form (1.6).

The expressions (3.9) and (3.10) have been obtained for an anisotropic body with arbitrary hardening and contain a number of particular cases, some of which we examine below.

4. Let us initially consider an isotropic body with arbitrary hardening. In this case the following expressions hold:

$$E^{ijnm} = \lambda g^{ij}g^{nm} + \mu(g^{in}g^{jm} + g^{im}g^{jn}) \qquad (4.1)$$

$$C_{nmij} = \frac{1}{2\mu}\left(g_{ni}g_{mj} - \frac{\nu}{1-\nu}g_{nm}g_{ij}\right) \qquad (4.2)$$

Substituting (4.1) into (3.10), we obtain after a number of manipulations (A_i^M are algebraic invariants of the tensor M)

$$E_*^{ijnm} = [\lambda^2 g^{ij}g^{nm}(A_1^M)^2 + 2\lambda\mu(g^{ij}M^{nm} + g^{nm}M^{ij})A_1^M + 4\mu^2 M^{ij}M^{nm}][2\mu A_2^M + \lambda(A_1^M)^2 - 1]^{-1} \qquad (4.3)$$

$$A_1^M = g^{\alpha\beta}M_{\alpha\beta}, \quad A_2^M = M_{\alpha\beta}M^{\alpha\beta} \qquad (4.4)$$

Therefore, in the case of an initially isotropic body with arbitrary hardening, the first form of the fundamental relationships has the form (2.1) taking (4.2) and (3.1) into account, while the second form has the form (3.9) with (4.1) and (4.3) taken into account. Further simplifications (particular cases) can be obtained if the kind of stress function is made specific.

As an illustration, let us examine the loading function corresponding to the Mises plasticity condition. We represent it as follows:

$$f = A_2'^{\sigma} - \varphi(\chi) \qquad (4.5)$$

$A_2'^{\sigma}$ is the second algebraic invariant of the stress tensor deviator, $\sigma'^{\alpha\beta}$ are components of the stress tensor deviator, χ is the hardening parameter. In conformity with (4.4), we can write the second algebraic invariant in terms of the components of the stress tensor deviator

$$A_2'^{\sigma} = \sigma'^{\alpha\beta}\sigma'_{\alpha\beta}.$$

Following [1, 2, 5], we take the work of the plastic strains as the hardening parameter, then

$$d\chi = \sigma^{ij} d\varepsilon_{ij}^{p}.$$

Comparing this with the expression (1. 5), we obtain $A^{ij} \equiv \sigma^{ij}$.

After transformations, we obtain the following expression from (4. 5) and (3. 2):

$$M_{nm} = \left(-\frac{1}{2} A_2'^{\sigma} \frac{d\varphi}{d\chi}\right)^{-1/2} \sigma'_{nm}$$

In this case we find K_{nmij} from (3. 1) and (3. 2), and E_*^{ijnm} from (4. 3), (4. 4) and (4.6):

$$K_{nmij} = 2\left(A_2'^{\sigma} \frac{\partial\varphi}{d\chi}\right)^{-1} \sigma'_{nm}\sigma'_{ij} \tag{4. 6}$$

$$E_*^{ijnm} = 8\mu\mu \Big/ \frac{d\varphi}{d\chi} \left(1 + 4\mu \Big/ \frac{d\varphi}{d\chi}\right)^{-1} \frac{\sigma'^{ij}\sigma'^{nm}}{A_2'^{\sigma}} \tag{4. 7}$$

Therefore, the fundamental relationships in the first form have the form (2. 1) for an initial isotropic body with a loading function in the form (4. 5), where the notation (4.2) and (4. 6) has been introduced, and have the form (3. 9) in the second form, where the notation (4. 1) and (4. 7) has been introduced. The fundamental relationships in the first form are presented in [5] in several other notations. The fundamental relationships in the second form for a loading function in the form (4. 5) are presented here as an illustration of the inversion of (3. 9) and (3. 10) for an arbitrary loading function.

In the case of a loading function in the form of (4. 5), the fundamental relationships can be given a still different form. To this end, let us introduce the following notation: E' is the tangential modulus on the uniaxial tension diagram, σ_u is the stress intensity. Following [5], let us assume that the work of the plastic strains is determined completely by the stress intensity. In this case, we can obtain by analogy with [5]

$$M_{km} = \frac{3}{2}\sqrt{\frac{1}{E} - \frac{1}{E'}}\,\frac{\sigma'_{km}}{\sigma_u}, \quad \sigma_u = \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'^{ij}} \tag{4. 8}$$

Substituting (4. 12) into (4. 3), we obtain after a number of transformations

$$E_*^{ijnm} = \frac{9}{2}\mu \frac{\sigma'^{ij}\sigma'^{nm}}{\sigma_u^2}\left(\frac{E}{E'} - 1\right)\left[1 + \nu + \frac{2}{3}\left(\frac{E}{E'} - 1\right)\right]^{-1}$$

We similarly obtain the fundamental relationships for other loading functions also.

REFERENCES

1. Ivlev, D. D. and Bykovtsev, G.I., Theory of a Hardening Plastic Body. "Nauka", Moscow, 1971.
2. Sedov, L.I., Mechanics of a Continuous Medium, Vol. 2, "Nauka", Moscow, 1970.
3. Guz', A. N., Stability of Three-Dimensional Deformable Bodies." Naukova Dumka", Kiev, 1971.
4. Kliushnikov, V. D., Bifurcation of the strain process and the concept of continuing loading. Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, № 5, 1972.
5. Kachanov, L. M., Principles of the Theory of Plasticity. "Nauka", Moscow, 1969.

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